LATTICE AND POSET VALUED FUZZY STRUCTURES
- theory and applications

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Overview (Applications)

- Lattice Valued Fuzzy Sets (Justification)
- Cuts and Cutworthy Approach
- Applications (examples):
- Dealing with concepts (social sciences)
- Distribution of species (biology)
- Making diagnosis (medicine)

Lattice-valued fuzzy sets

- Fuzzy set

$$\mu : A \rightarrow L$$

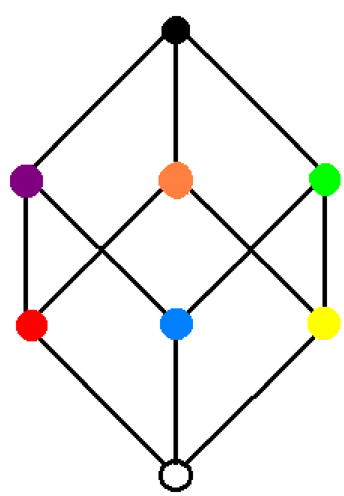
L is a complete lattice.

- *For every* $p \in L$, *p*-cut or a *p*-level (subset) of (A,μ):

$$\mu_p = \{x \in A \mid \mu(x) \geq p\}.$$

Lattice-valued fuzzy sets (justification) and cut sets (levels)

Lattice co-domain



- $\mu_L=\{\mu_p | p \in L\}$ is a set of all cuts of a lattice valued fuzzy set μ.
- *Cutworthy approach:*

  A fuzzy set satisfies property P if and only if all the cuts (crisp structures) satisfy the corresponding crisp property P.

- Cutworthy approach has several advantages in application and research of fuzziness.
- It enables parallel investigation of fuzzy structures as functions as well as collections of crisp structures (usually well known and their properties can be successfully transferred to the new, fuzzy structure).
- This approach is widely applied in investigation of fuzzy relations, fuzzy algebraic structures , different applications etc.

Applications

- Dealing with concepts (social sciences) - Formal concept analysis
- Distribution of species (biology) - Fuzzy correspondences
- Making diagnosis (medicine)- Fuzzy correspondences and fuzzy intuitionistic correspondences

Formal Concept Analysis

- Formal concept analysis is used to identify similarities in data (by using lattices).
- Use in data analysis and knowledge processing.
- Developed at TU Darmstadt, Germany (R. Wille, B. Ganter).

- B. Ganter, R. Wille: Formal Concept Analysis - Mathematical Foundations, Springer, 1999.

A formal context $K=(G,M,I)$ consists of two sets G and M and a relation (correspondence) I between G and M. The elements of G are called objects and the elements of M attributes of the context.

For a set $A \subseteq G$ of objects, we define:
$A' = \{m \in M \mid (g,m) \in I \text{ for all } g \in A\}$
(the set of attributes common to the objects in A).

Similarly, for a set B of attributes, we define:
$B' = \{g \in G \mid (g,m) \in I \text{ for all } m \in B\}$
(the set of objects which have all the attributes in B).

- A formal concept of the context *(G,M,I)* is a pair *(A,B)*, with $A \subseteq G$, $B \subseteq M$, $A' = B$ and $B' = A$.
- *A* is then the extent and *B* the intent of the concept *(A,B).*
- *B(G,M,I)* is the set of all concepts of the context *(G,M,I).*

- If *(A,B)* and *(C,D)* are two concepts of the same contest, then *(A,B)* is a **subconcept** of *(C,D)* if $A \subseteq C$ or $D \subseteq B$.
- Then, also *(C,D)* is a **superconcept** of *(A,B),* and we write $(A,B) \leq (C,D)$. The relation $\leq$ is called the hierarchical order of the concepts.
- *B (G,M,I)* is a lattice under this ordering relation and is called the **concept lattice.**

- Role of lattices in formal concept analysis is twofold:
- Set of all concepts of the context is a *lattice*
- Many valued contexts are fuzzy sets and they can be consider as a *lattice* valued fuzzy sets.

Mushrooms example

- *G*={Boletus edulis, Agaricus xanthoderma, Morchella esculenta}.
- *M*={saprophyte, edible, basidiomyceta}.
- *I* is the relation "mushroom species *x* has property *y*".

	saprophyte	edible	basidiomyceta
Boletus edulis		X	X
Agaricus xanthoderma	X		X
Morchella esculenta	X	X	

- {Boletus edulis}'={edible, basidiomyceta}
- {edible, basidiomyceta}'= {Boletus edulis}

Therefore,

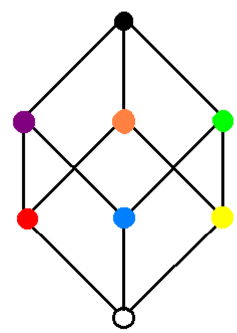
- ({Boletus edulis}, {edible, basidiomyceta}) is one concept.

- {Boletus edulis, Agaricus xanthoderma}'={basidiomyceta}
- {basidiomyceta}'= {Boletus edulis, Agaricus xanthoderma}

Therefore,

- ({Boletus edulis, Agaricus xanthoderma}, {basidiomyceta}) is another concept.

Concept lattice



- Many valued contexts – fuzzy sets
- A many valued context *(G,M,W,I),*

  where *G*,*M* and *W* are sets and $I \subseteq G\text{x}M\text{x}W$ is a ternary relation.

  G-objects

  M-attributes

  W-attribute values

(G,M,W,I), $I \subseteq G\text{x}M\text{x}W$

I-ternary relation.

Instead of *I,* a special mapping

$$J: G\text{x}M \rightarrow W$$

is considered in order to obtain a fuzzy relation, where *W* is an ordered set of attribute values.

Many valued contexts (continuation of mushroom example)

$W=\{0,s,m,g,1\}$

0 - an object does not has an attribute at all

s - an object has an attribute to a small extent

m - an object has an attribute to a middle extent

g - an object has an attribute to a great extent

1 - an object has an attribute (completely).

Ordered set $0<s<m<g<1$

	saprophyte	edible	basidiomyceta
Boletus edulis	s	1	1
Agaricus xanthoderma	1	s	1
Morchella esculenta	1	m	0

m-cut of this fuzzy relation

	saprophyte	edible	basidiomyceta
Boletus edulis	0	1	1
Agaricus xanthoderma	1	0	1
Morchella esculenta	1	1	0

Applications in Biology

Construction:

If ρ is a correspondence from a set *A* to a set *B*, then a quasi-ordering relation θ on *B* could be defined in the following way:

$(x,y)\in\theta \longleftrightarrow (\forall z\in A)((z,x)\in\rho \to (z,y)\in\rho)$ (and a similar relation on *A*).

Equivalence relation on *B*:

$(x,y)\in\varepsilon \longleftrightarrow (x,y)\in\theta \wedge (y,x)\in\theta.$

Ordering relation on the set of equivalence classes:

$X\leq Y \longleftrightarrow (\exists x\in X)(\exists y\in Y)(x,y)\in\theta.$

- Let $R:S^2\to L$ be an *L*-valued fuzzy relation, where $(L, \wedge,\vee)$ is a complete lattice with the top element 1 and the bottom element 0.

Then the relation *R* is:

- **reflexive** if $R(x,x) =1$ for all $x \in S$;
- **symmetric** if $R(x,y) = R(y,x)$, for all $x,y \in S$;
- **transitive** if $R(x,y)\wedge R(y,z)\leq R(x,z)$, for all $x,y,z \in S$;
- **antisymmetric** if $R(x,y)\wedge R(y,x) = 0$, for all $x,y \in S$, $x \neq y$.

The fuzzy relation *R* is

- -fuzzy **similarity** relation if it is reflexive, symmetric and transitive;
- -fuzzy **quasi-ordering** relation if it is reflexive and transitive;
- -fuzzy **ordering** relation if it is reflexive, antisymmetric and transitive.

- Analogous fuzzy construction is given for a special type of fuzzy correspondences which appear in biogeography.
- From this correspondence a quasi-ordering relation is obtained, as well as the corresponding fuzzy similarity relation and fuzzy ordering relation on the set of similarity classes.

- Problems of determining connections of different biogeographical areas according to recorded species have been considered.
- The results have been applied (by A. Tepavcevic, A.Vujic) to determine connections of biogeographical territories in the Balkan Peninsula, according to different species of Genus *Cheilosia* Meigen 1822 and related genera (Diptera: Syrphidae).

- Let V be a set of different species and N a set of biogeographical territories.
- First, the following correspondence on $V \times N$ is observed:

  $(x,y) \in \rho$ if and only if species x could be found on territory y.
- Starting with this correspondence, a connection of territories in N, according to the species living there, can be determined. However, this approach is only a particular case (level 1) in fuzzy consideration that follows.

- In the fuzzy case, a zoogeographical significance of species is considered. All species from *V* are divided into 6 classes, according to their zoogeographical significance (from species with very little significance (1) to very important endemic species (6))

- A suitable fuzzy correspondence $\rho: V \times N \rightarrow L$ is defined by

- $\rho(x,y)=z \in \{1,2,3,4,5,6\}$ if and only if the species *x* with the significance *z* was found on the territory *y* and
- $\rho(x,y)=0$ if and only if the species *x* was not found on the territory *y*.

(10 territories and 81 species).

Pollen example:

Basic ideas:

- Consider a fuzzy relation *R* on
- the set of all different species of pollen grains *S* defined by:
- *R(x,y)=p*

if and only if

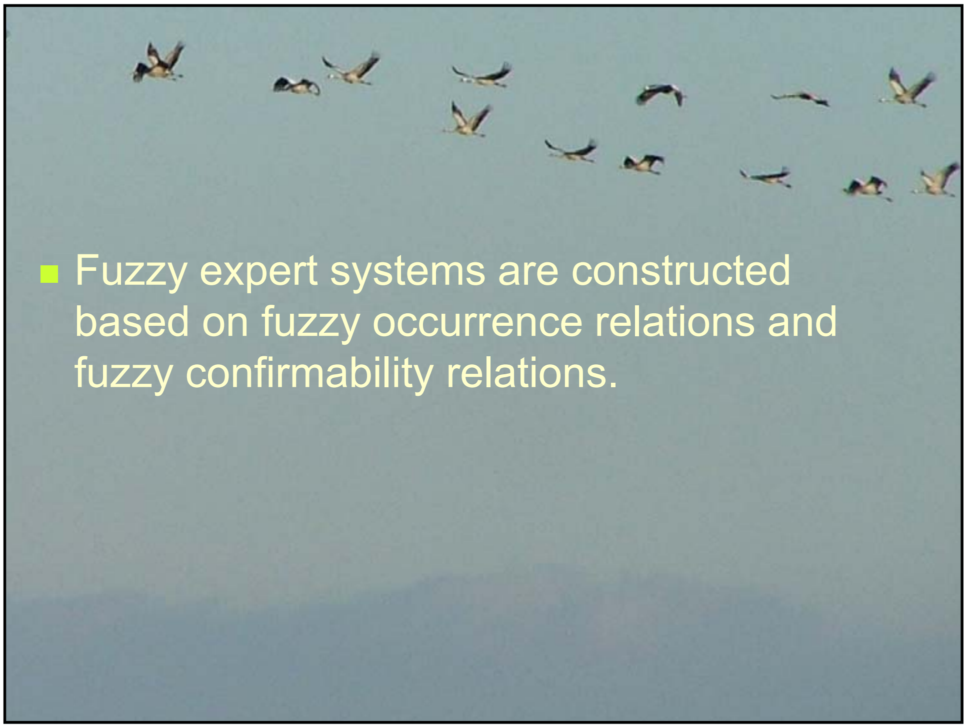
"pollen *x* appears before (in period of year) the pollen *y"* with grade *p*

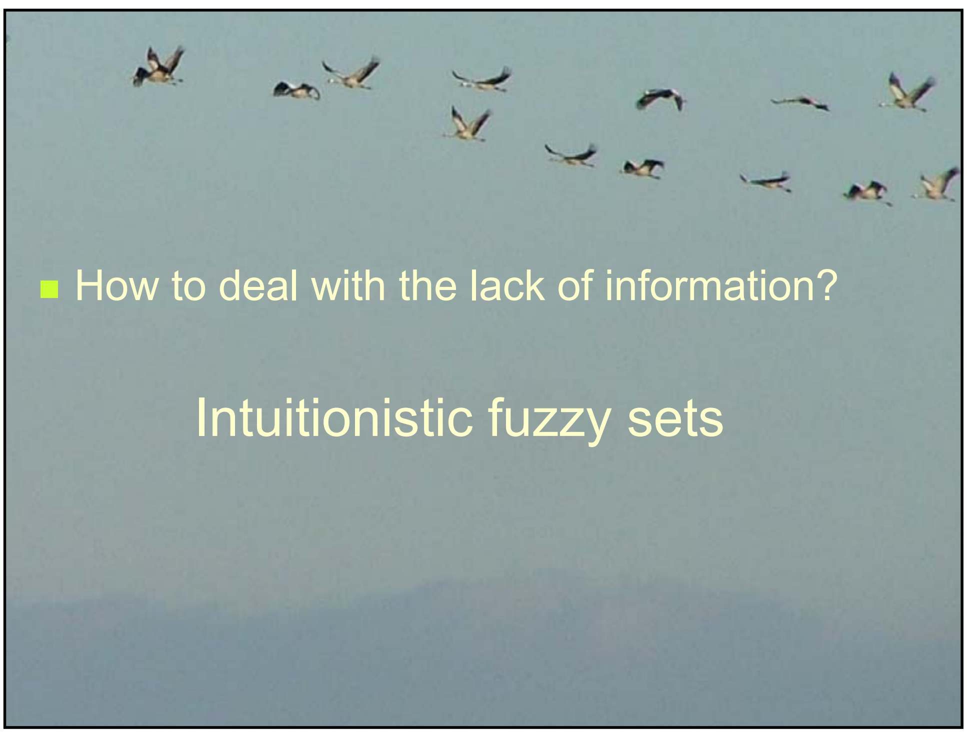
Example of an application in Medicine

- Problem of making the diagnosis according to the symptoms and signs:

Occurrence relations and **confirmability relations** are used (fuzzy correspondences): mappings from *S*×*D* to an ordered structure), where *S* is the set of all symptoms and signs and *D* the set of all possible diagnosis.

- The occurrence relations represent the grade of occurrence of symptoms related with different diagnosis and the confirmability relations represent the grade to which a diagnosis is confirmed by a symptom.


Fuzzy expert systems are constructed based on fuzzy occurrence relations and fuzzy confirmability relations.


How to deal with the lack of information?
Intuitionistic fuzzy sets

- Let X be a nonempty set, and μ and ν be two functions from X to [0,1], such that for all $x \in X$,

$$0 \leq \mu(x) + \nu(x) \leq 1.$$

  - $\mu(x)$ is the degree of membership
  - $\nu(x)$ is the degree of non-membership.
- By the original definition of Atanassov:

  an **intuitionistic fuzzy set** is an object of the form:

$$A=\{(x, \mu(x), \nu(x)) | x \in X\}.$$

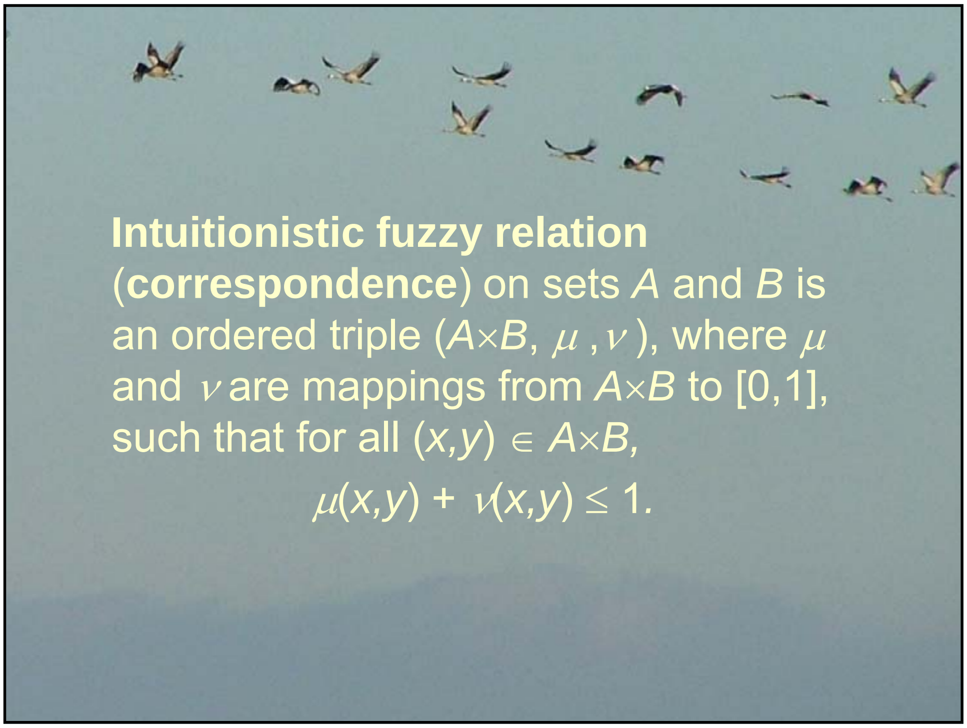
- It is usually considered in a form of an ordered triple:

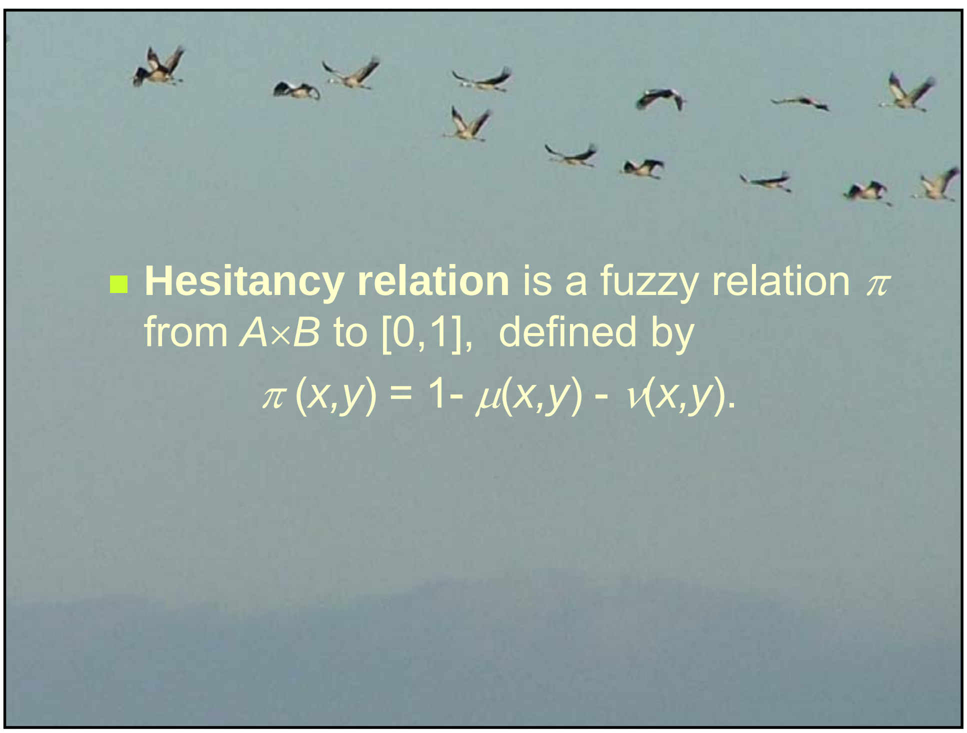
  $A = (X, \mu, \nu)$, where X, μ and ν are as above.

- **Hesitancy margin** (intuitionistic fuzzy index) is a fuzzy set connected with an intuitionistic fuzzy set.
- The hesitancy margin π of an element x in a set A, is defined by

$$\pi(x) = 1 - \mu(x) - \nu(x).$$

- The hesitancy margin π present level of uncertainty (indeterminacy).


Intuitionistic fuzzy relation (correspondence) on sets A and B is an ordered triple $(A \times B, \mu, \nu)$, where μ and ν are mappings from $A \times B$ to [0,1], such that for all $(x,y) \in A \times B$,
$$\mu(x,y) + \nu(x,y) \leq 1.$$


- **Hesitancy relation** is a fuzzy relation π from $A \times B$ to [0,1], defined by
$$\pi(x,y) = 1 - \mu(x,y) - \nu(x,y).$$

- The hesitancy relation represents a degree of indeterminacy about the relationship of elements from sets *A* and *B*.

- Moreover, it gives a possibility of changing the values μ and ν according to the influx of information and evaluation of the (medical) knowledge.

A simple theoretical example:

Let $S = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\}$ be a set of symptoms,

and let the possible diagnosis are $D=\{d_1, d_2, d_3, d_4, d_5, d_6\}$.

Intuitionistic confirmability relation

C_p	d_1	d_2	d_3	d_4	d_5	d_6
s_1	0.7	0.6	0	0.1	0	0.1
s_2	0.1	0.9	0	0.1	0.3	0
s_3	0.3	0.1	0.9	0.1	0	0.1
s_4	0.1	0	0.1	0.1	0.9	0.1
s_5	0.7	0.2	0.3	0.8	0.8	0.7
s_6	0.3	0.2	0.7	0.1	0.1	0.9
s_7	0.2	0	0.3	0.8	0.4	0
s_8	0.5	0.3	0.8	0.1	0	0

C_n	d_1	d_2	d_3	d_4	d_5	d_6
s_1	0.2	0.3	1	0.8	1	0.7
s_2	0.8	0.1	1	0.8	0.6	0.9
s_3	0.5	0.8	0.1	0.8	0.9	0.8
s_4	0.7	0.8	0.5	0.5	0.1	0.8
s_5	0.2	0.8	0.6	0.2	0.2	0.2
s_6	0.5	0.7	0.2	0.8	0.8	0.1
s_7	0.6	1	0.5	0.1	0.5	1
s_8	0.4	0.6	0.2	0.7	0.9	0.9

Intuitionistic occurrence relation

O_p	d_1	d_2	d_3	d_4	d_5	d_6
s_1	1	0.8	0	0.2	0	0.5
s_2	0.2	0.5	0	0.2	0.5	0
s_3	0.5	0.3	1	0.1	0	0.1
s_4	0.1	0	0.2	0.5	0.9	0.4
s_5	0.9	0.4	0.3	1	0.8	1
s_6	0.5	0.3	0.8	0.1	0.1	1
s_7	0.3	0	0.5	1	0.7	0
s_8	0.7	0.5	0.9	0.5	0	0

O_n	d_1	d_2	d_3	d_4	d_5	d_6
s_1	0	0.1	0.9	0.5	1	0.1
s_2	0.5	0.3	1	0.7	0.5	1
s_3	0.2	0.6	0	0.8	1	0.8
s_4	0.8	1	0.3	0.5	0.1	0.3
s_5	0	0.3	0.5	0	0	0
s_6	0.2	0.6	0.1	0.8	0.8	0
s_7	0.5	1	0.4	0	0.2	1
s_8	0.1	0.3	0.1	0.4	1	1

Hesitancy relations

H_o	d_1	d_2	d_3	d_4	d_5	d_6
s_1	0	0.1	0.1	0.3	0	0.4
s_2	0.3	0.2	0	0.1	0	0
s_3	0.3	0.1	0	0	0	0.1
s_4	0.1	0	0.5	0	0	0.3
s_5	0.1	0.3	0.2	0	0.2	0
s_6	0.3	0.1	0.1	0.1	0.1	0
s_7	0.2	0	0.1	0	0.1	0
s_8	0.2	0.2	0	0.1	0	0

H_c	d_1	d_2	d_3	d_4	d_5	d_6
s_1	0.1	0.1	0	0.1	0	0.2
s_2	0.1	0	0	0.1	0.1	0.1
s_3	0.2	0.1	0	0.1	0.1	0.2
s_4	0	0.1	0.4	0.4	0	0.1
s_5	0.1	0	0.1	0	0	0.1
s_6	0.2	0.1	0.1	0.1	0.1	0
s_7	0.2	0	0.2	0.1	0.1	0
s_8	0.1	0.1	0	0.2	0.1	0.1

- Analogous notions in lattice valued settings are also used:

  Lattice valued intuitionistic fuzzy sets and relations.